Malaysian Journal of Mathematical Sciences 17(3): 487–508 (2023)
https://doi.org/10.47836/mjms.17.3.15

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

The Implicit Multistep Block Method with An Off-step Point for Initial Value Problems of Neutral Delay Volterra Integro-differential Equations

Ismail, N. I. N.*[1] and Majid, Z. A.[1,2]

[1]*Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
[2]*Department of Mathematics and Statistics, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*

E-mail: inshirahismail8996@gmail.com
**Corresponding author*

Received: 13 February 2023
Accepted: 29 August 2023

Abstract

The aim of this manuscript is to solve the initial-value problems of neutral delay Volterra integro-differential equations with constant or proportional delays. Hence, a proposed hybrid technique named as an implicit multistep block method with an off-step point (1OBM4) is formulated for the numerical solution of NDVIDE. A LMM associated with an off-point is known as hybrid LMM. The proposed technique, 1OBM4, attempts to solve the problem synchronously in a block manner. Moreover, a Taylor expansion is implemented to develop 1OBM4 in predictor-corrector mode. Two different approaches are presented in order to solve both integral and differential parts of the problem. Some analyses on 1OBM4 are considered in terms of order and convergence of the method. A stability polynomial is also obtained for the stability regions to be constructed. In the last section, some numerical results are demonstrated to show the applicability of 1OBM4 in solving NDVIDE with constant or proportional delays.

Keywords: constant delay; constant step size; implicit multistep block method; neutral delay Volterra integro-differential equations; off-step point; proportional delay.

1 Introduction

Recently, the applications of neutral delay Volterra integro-differential equation (NDVIDE) have shown significant advances in biological and physical sciences. Many analytical and numerical solutions have been proposed by different authors to resolved the problems yet there is still a huge gap in the application of linear multistep method (LMM). In recent times, NDVIDE has received great attention as a proper model for time delay problems, particularly in the engineering and biological sector. The pantograph delay is a type of time delay system, however unlike other time delay systems, it operates proportionally. It gets its name from the pantograph on a train, a device located on the roof of an electric train, [12].

In this contribution, a role that may be played by NDVIDE will be indicated in modeling some cell development phenomena that exhibit a delay in their response to events. Every scientific discipline considers the challenge of developing a mathematical model to explain the behaviour of a system as characterized by a time series of observations to be central, [5]. The time delay issue should be taken into account from the beginning of the design process since it is one of the variables that affect the dynamic. A neutral type of DVIDE typically emerges in the modeling of linked oscillatory systems, where the oscillators are connected and allow for the transfer of energy between them, as a comparison example. The following time delay is used to realistically depict the model,

$$\begin{aligned} y'(x) &= f\big(x, \rho_0 y(x)\big) + \int_{x-\tau_i}^{x} K\Big(x, \rho_0 y(x), \rho_1 y(x-\tau_i), \rho_2 y'(x-\tau_i)\Big), && x \geq x_0,\\ y(x) &= \phi(x), && x \leq x_0,\\ y'(x) &= \phi'(x), && x \leq x_0, \end{aligned} \tag{1}$$

where $y(x-\tau_i)$ and $y'(x-\tau_i)$, for $0 \leq i \leq n$, are the expressions of delay solutions and its derivative, τ_i is known as the delay, $f\big(x, \rho_0 y(x)\big)$ is the function containing either the expression of delay solution or its derivative, while $K\Big(x, \rho_0 y(x), \rho_1 y(x-\tau_i), \rho_2 y'(x-\tau_i)\Big)$ is a function called the kernel or the nucleus of the integral equation, [28]. Both $f\big(x, \rho_0 y(x)\big)$ and $K\Big(x, \rho_0 y(x), \rho_1 y\big(x-\tau_i\big), \rho_2 y'\big(x-\tau_i\big)\Big)$ need to be continuous. Since just a constant value of delay is taken into account, equation (1) is referred to as a constant form of NDVIDE. In addition to continuous NDVIDE, a proportional delay (another name for the pantograph equation) type is also crucial for the advancement of the industry. The following is a general illustration of the pantograph equation that NDVIDE is modeling,

$$\begin{aligned} y'(x) &= f\big(x, \rho_0 y(x)\big) + \int_{qx}^{x} K\Big(x, \rho_0 y(x), \rho_1 y(qx), \rho_2 y'(qx)\Big), && x \geq x_0,\\ y(x) &= \phi(x), && x \leq x_0, \end{aligned} \tag{2}$$

where $y(x) = \phi(x)$ is the given initial function and $0 \leq q \leq 1$, [13]. According to [2], the initial function, $\phi(x)$ is defined in $[\rho, x_0]$, as shown below,

$$\rho = \min_{1\leq i\leq n} \big\{ \min_{x\geq x_0} (x-\tau_i) \big\}, \tag{3}$$

since for some $x \geq x_0$, $x - \tau_i < x_0$.

2 Development of Method

Numerous scholars have found numerical solutions to the initial condition of NDVIDE. Jackiewicz has solved a number of neutral delay problems, and one of his monographs discusses the multistep Adams-Moulton approach for solving NDVIDE problems, [15]. The general convergence theorem is the subject of an early investigation, and the order in which these methodologies converge is studied. Obviously, the current approach can be added to any multistep process to address the issues for future research. [9] have studied the explicit and implicit continuous Runge-Kutta methods for solving the NDVIDE with constant delay. The numerical outcomes demonstrate the method's ability to produce a continuous approximation with a global error limited by a minor multiple of the selected error tolerance. Subsequently, [10] has created a novel polynomial collocation solution to NDVIDE with constant delay, and he has conducted a study of the global convergence and local superconvergence aspects of this solution. Since the monograph solely focused on theoretical findings, the stability characteristics of the collocation method are instead examined elsewhere. [35] has focussed on the stability of numerical techniques for linear NDVIDE systems. The system must meet a requirement in order to be asymptotically stable. Furthermore, it has been demonstrated that all linear-methods with $\theta \in \left(\frac{1}{2}, 1\right]$ and A-stable BDF methods maintain the delay-independent stability of their exact solutions. Later, [8] has reported recent advancements and some unresolved issues in the numerical solution of NDVIDE with proportional delay. Related functional equations are explained, together with theoretical and computational features of collocation methods used to solve them. Despite all the theories, due to the method's space constraints, a more thorough treatment of the precise theoretical and computational issues relevant to the NDVIDE is not given.

Following that, [29] have examined the analytical and numerical stability of NDVIDE and neutral delay partial differential equations (NDPDE) in addition to [35]. The test equations that demonstrate the preservation of the delay-independent stability are considered. To support the theoretical findings, some numerical experiments are presented. The asymptotic stability of exact and discrete solutions to neutral multidelay integro-differential equations was the subject of [33]'s study in the same year. The equations are resolved using modifications of the conventional Runge-Kutta (RK) and LMM. It is found that, under the right conditions, these two groups of numerical techniques preserve the continuous systems' stability. Later, according to [31], the approximate solution of a linear NDVIDE with three different types of equations might be discovered using the Galerkin method with the Bernsien polynomial as the basis function. Some key formulae for the Bernstein polynomial, which is crucial to numerical calculations, have been obtained in relation to the derivative of orthogonal polynomials. Following that, [22] have proposed a numerical method based on a spectrum approach for the solution of DVIDE issues of the neutral type. The approach is easily adaptable to neutral-type nonlinear DVIDE. The proportional type of NDVIDE is then resolved under the initial conditions by [30] using a collocation method based on Laguerre polynomials. The issue is simplified to a set of algebraic equations by using the Laguerre polynomials, matrix operations, and evenly spaced collocation sites. Solving this system yields the coefficients needed to arrive at an approximation of the solution to the original issue. A method error estimation is also introduced using the residual function. The approximation is corrected with respect to the estimated error function.

Later, [27] thought about a class of abstract neutral integro-differential inclusions in Hilbert spaces with infinite delay the following year. By proving the Bohnenblust-Karlin fixed point theorem, he has found solutions to the issues. Then, new specific conditions for the asymptotic stability and boundedness of solutions to nonlinear Volterra integro-differential equations of first order with constant retardation have been constructed by [25]. The analysis is based on the successful construction of suitable Lyapunov-Krasovskii functionals (LKF). The results in the paper are new, and they have improved and completed the method taken from previous literature. In the next year, the dissipativity and stability of the theoretical solutions of a class of nonlinear RDVIDE with mixed delays are taken into account by [32]. A generalized Halanay inequality is given which plays an important role in the study of dissipativity and stability of IDE. Then, the generalized Halanay inequality is applied to the dissipativity and stability of the theoretical solution of RDVIDE. Finally, the results are provided to demonstrate the effectiveness and advantages of the theoretical results.

Recent research by [3] on the asymptotic behaviors of NDVIDE issue solutions led to the discovery of unique, necessary criteria for their Lyapunov method establishment. The differential transformation method (DTM) is then used to resolve a specific case of the NDVIDE under consideration. Then, [14] provided an explicit third-order block multistep method, where the method is obtained using the Taylor series, to solve constant type NVDIDE and RVDIDE problems. It appears that the technique can be descended from the method. Following [25], a more suitable LKF has been constructed by [26] where they have investigated the integrability of the norm and boundedness of solutions. A contribution to the topic of the paper and relevant literature is provided from the numerical examples for the uniformly asymptotically stability of zero solution as well as integrability and boundedness of solutions. [17] have more recently used the Taylor collocation method to numerically solve a k^{th}-order NDVIDE with constant delay. The approach is simple to use, convergent, and accurate. It will be possible to perform more research on this type of issue by applying the results to a system of k^{th}-order NDVIDE. The majority of earlier authors have similarly prioritized the analytical method over the numerical method. However, a number of one-step and multistep methods have been presented by researchers for the numerical solutions of NDVIDEs, yet none of them have proposed a hybrid multistep block technique. The purpose of this study is to extend the previous multistep method's work to a multistep block with an off-point.

3 Methodology

This section will provide a detailed description of the block method with an off-step point's development for the numerical solution of NDVIDE with constant or proportional delay. A specific explanation on formulation, order and convergence will be depicted thoroughly. The Adam-Bashforth predictor mechanism is taken into consideration during the development of implicit multistep block method with an off-step point. The idea of the proposed method is taken from [20] and has been modified. The two-point block with the following off-step point is taken into account,

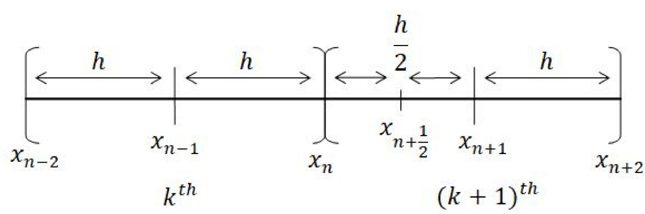


Figure 1: Two-point block with an off-point.

The interval $[a, b]$ is subdivided into a sequence of blocks, where each block contains two-point including an off-point. These two points will be evaluated simultaneously where the first block operated as initial solutions for the next block. The process will continue for subsequent iterations in other blocks until the interval's completion. The hybrid block technique has the benefit of minimizing the number of computational steps required.

3.1 Formulation of method

In order to solve NDVIDE with constant or proportional delay, the method has been improved to become more straightforward and effective. Given below is the linear difference operator, L associated with an off-step point, as indicated in [18],

$$L\big[y(x);h\big] = \sum_{j=0}^{k}\Big[\alpha_j y\big(x+jh\big) - h\beta_j y'\big(x+jh\big)\Big] - h\beta_v y'(x+vh), \tag{4}$$

where the Taylor terms for $y(x+jh)$ and $y_0(x+jh)$ will be extended such that,

$$L\big[y(x);h\big] = C_0 y(x) + C_1 h y^{(1)}(x) + \cdots + C_q h^q y^{(q)}(x). \tag{5}$$

Local truncation error, also known as local discretization error, is the term $C_{q+1}h^{q+1}y^{(q+1)}(x)$ that emerges after the truncation. In order to derive first predictor formula of the proposed method, a k-step hybrid formula based on equation (4) is expanded as shown below to formulate the implicit hybrid multistep block method with an off-step point,

$$y_{n+4} + \alpha_0 y_{n+3} = h\sum_{i=0}^{3}\beta_i y'(x+ih). \tag{6}$$

By employing Taylor series, extending each $y(x)$ and $y'(x)$,

$$\begin{aligned}
y(x+4h) &= y(x) + 4hy'(x) + 8h^2y''(x) + \frac{32}{3}h^3y'''(x) + \frac{32}{3}h^4y^{(4)}(x),\\
y(x+3h) &= y(x) + 3hy'(x) + \frac{9}{2}h^2y''(x) + \frac{9}{2}h^3y'''(x) + \frac{27}{8}h^4y^{(4)}(x),\\
y'(x+h) &= y'(x) + hy''(x) + \frac{1}{2}h^2y'''(x) + \frac{1}{6}h^3y^{(4)}(x),\\
y'(x+2h) &= y'(x) + 2hy''(x) + 2h^2y'''(x) + \frac{4}{3}h^3y^{(4)}(x),\\
y'(x+3h) &= y'(x) + 3hy''(x) + \frac{9}{2}h^2y'''(x) + \frac{9}{2}h^3y^{(4)}(x).
\end{aligned} \tag{7}$$

Every derivative term is collected and formed,

$$\begin{aligned}
&\left[hy'(x) + \frac{7}{2}h^2y''(x) + \frac{37}{6}h^3y'''(x) + \frac{175}{24}h^4y^{(4)}(x)\right]\\
&\quad= \Big[\beta_0+\beta_1+\beta_2+\beta_3\Big]hy'(x) + \Big[\beta_1+2\beta_2+3\beta_3\Big]h^2y''(x) + \left[\frac{1}{2}\beta_1+2\beta_2+\frac{9}{2}\beta_3\right]h^3y'''(x)\\
&\qquad+\left[\frac{1}{6}\beta_1+\frac{4}{3}\beta_2+\frac{9}{2}\beta_3\right]h^4y^{(4)}(x),
\end{aligned} \tag{8}$$

as equation (7) is substituted into equation (6). The values of the β_i coefficients are calculated after equating the left and right hand sides.

$$\begin{aligned} 1 &= \beta_0 + \beta_1 + \beta_2 + \beta_3, \\ \frac{7}{2} &= \beta_1 + 2\beta_2 + 3\beta_3, \\ \frac{37}{6} &= \frac{1}{2}\beta_1 + 2\beta_2 + \frac{9}{2}\beta_3, \\ \frac{175}{24} &= \frac{1}{6}\beta_1 + \frac{4}{3}\beta_2 + \frac{9}{2}\beta_3, \end{aligned} \tag{9}$$

with $\beta_0 = -\frac{3}{8}, \beta_1 = \frac{37}{24}, \beta_2 = -\frac{59}{24}$ and $\beta_3 = \frac{55}{24}$. By letting $n = n-3$, the first point predictor formula, y^p_{n+1} is obtained below,

$$y^p_{n+1} = y_n + \frac{h}{24}\Big(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\Big). \tag{10}$$

The derivations of the first point predictor ($y^p_{n+\frac{1}{2}}$) and the second point predictor (y^p_{n+2}) are similar to the earlier implementations where the two original forms are displayed as follows,

$$y_{n+\frac{5}{2}} + \alpha_0 y_{n+2} = h\sum_{i=0}^{3}\beta_i y'(x+ih), \tag{11}$$

for $y^p_{n+\frac{1}{2}}$, and,

$$y_{n+6} + \alpha_0 y_{n+4} = h\sum_{i=1}^{4}\beta_i y'(x+ih), \tag{12}$$

for y^p_{n+2}. Alongside, the first, y^c_{n+1}, and second point corrector formula, y^c_{n+2}, are given by,

$$y_{n+3} + \alpha_0 y_{n+2} = h\left[\sum_{i=1}^{3}\beta_i y'(x+ih) + \sum_{v=\frac{5}{2}}^{\frac{5}{2}}\beta_v y'(x+vh)\right], \tag{13}$$

and,

$$y_{n+5} + \alpha_0 y_{n+3} = h\left[\sum_{i=2}^{2}\beta_i y'(x+ih) + \sum_{i=4}^{5}\beta_i y'(x+ih) + \sum_{v=\frac{7}{2}}^{\frac{7}{2}}\beta_v y'(x+vh)\right]. \tag{14}$$

After applying the same procedure, the proposed method is obtained as shown below,

$$\begin{aligned} y^p_{n+1} &= y_n + \frac{h}{24}\Big(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\Big), \\ y^p_{n+\frac{1}{2}} &= y_n + \frac{h}{384}\Big(25f_{n+1} + 197f_n - 37f_{n-1} + 7f_{n-2}\Big), \\ y^p_{n+2} &= y_n + \frac{h}{3}\Big(27f_n - 44f_{n-1} + 31f_{n-2} - 8f_{n-3}\Big), \\ y^c_{n+1} &= y_n + \frac{h}{6}\Big(f_{n+1} + 4f_{n+\frac{1}{2}} + f_n\Big), \\ y^c_{n+2} &= y_n + \frac{h}{27}\Big(10f_{n+2} + 27f_{n+1} + 16f_{n+\frac{1}{2}} + f_{n-1}\Big). \end{aligned} \tag{15}$$

where the given name is, 1OBM4, which will be employed to solve NDVIDE with constant or proportional delay types.

3.2 Order and error constant of method

According to [16],

Definition 3.1. *The hybrid method* (15) *is said to be of order* p *if,* $C_0 = C_1 = \cdots = C_q = 0$ *and* $C_{q+1} \neq 0$ *is called as an error constant where* $q = 2, 3 \ldots$.

Following [16], the order for 1OBM4 is determined as shown below,

$$C_0 = \sum_{j=0}^{k} \alpha_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j - \sum_{j=1}^{1} v_j \beta_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \frac{1}{2!}\left[\sum_{j=1}^{k} j^2\alpha_j - 2\left(\sum_{j=1}^{k} j\beta_j + \sum_{j=1}^{1} v_j\beta_j\right)\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_3 = \frac{1}{3!}\left[\sum_{j=1}^{k} j^3\alpha_j - 3\left(\sum_{j=1}^{k} j^2\beta_j + \sum_{j=1}^{1} v_j^2\beta_j\right)\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_4 = \frac{1}{4!}\left[\sum_{j=1}^{k} j^4\alpha_j - 4\left(\sum_{j=1}^{k} j^3\beta_j + \sum_{j=1}^{1} v_j^3\beta_j\right)\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_5 = \frac{1}{5!}\left[\sum_{j=1}^{k} j^5\alpha_j - 5\left(\sum_{j=1}^{k} j^4\beta_j + \sum_{j=1}^{1} v_j^4\beta_j\right)\right] = \begin{bmatrix} -\frac{1}{2880} \\ -\frac{1}{40} \end{bmatrix}.$$

Hence, 1OBM4 is of order four with error constant, $C_5 = \left[-\dfrac{1}{2880}, \quad -\dfrac{1}{40}\right]^T$. The significance of identifying the method's order is to identify whichever order is required to attain the requisite accuracy.

3.3 Convergence of method

Based on [1], the LMM is said to be converged if both conditions of consistency and zero-stability are satisfied.

Definition 3.2. *A LMM is said to be consistent if it is of order* $p \geq 1$ *and satisfies,*

$$\sum_{j=0}^{k} \alpha_j = 0, \quad \textit{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j. \tag{16}$$

The proposed 1OBM4 has been proven to have order $4 = p \geq 1$. Considering the second condition in Definition (3.2),

$$C_0 = \sum_{j=0}^{k} \alpha_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

followed by,

$$\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore, the proposed method is consistent after Definition (3.2) is proven. In any real-world problem, the determination of consistency is important since it is the key performance in predicting the proposed method to be better.

Definition 3.3. *A LMM is considered to be zero-stable if the root of the first characteristic polynomial below is not greater than one, given by,*

$$\begin{aligned} \rho(\xi) &= \left| \sum_{j=0}^{k} A_j \xi^{k-j} \right| = \left| \sum_{j=0}^{1} A_j \xi^{1-j} \right|, \\ &= \left| A_0 \xi^1 + A_1 \xi^0 \right|, \\ &= \left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xi^1 + \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \xi^0 \right|, \\ &= \left| \begin{bmatrix} \xi & 0 \\ 0 & \xi \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right|, \\ &= \left| \begin{bmatrix} \xi & -1 \\ 0 & \xi - 1 \end{bmatrix} \right|, \\ &= \xi(\xi - 1), \end{aligned} \tag{17}$$

where the values of ξ_j are no greater than one,

$$\xi(\xi - 1) = 0, \qquad \xi = 0, 1.$$

Since the obtained roots, ξ_i, are not greater than one, the proposed 1OBM4 is shown to be zero-stable. $\rho(\xi)$ satisfies the root condition for the first and second point formula where it is strongly stable. All roots of $\rho(\xi)$ are on the unit circle, hence they are stable. Therefore, 1OBM4 is concluded to be converged as both conditions for consistency and zero-stable are proven. Other than definition from [1], the LMM is also said to be converged by [18] if,

$$\lim_{h \to 0} y_i = y^*(x_i), \tag{18}$$

where y_i is the derived approximate solution in equation (15) and $y^*(x_i)$ is the exact solution given in equation (31) whereby $x \in [a, b]$. The applicability of any proposed method will be discovered by analyzing the convergence for any intended method, in any differential problems. If the approximate solution moved closer to the exact solution, as mentioned in equation (18), the method is said to be converged. Referring to a Lipschitz condition below,

$$\begin{aligned} &\Bigg| \left(f\big(x, \rho_0 y^*(x)\big) + \int_{x-\tau_i}^{x} K\Big(x, \rho_0 y^*(x), \rho_1 y^*(x - \tau_i), \rho_2 {y^*}'(x - \tau_i)\Big) dx \right) \\ &\quad - \left(f\big(x, \rho_0 y(x)\big) + \int_{x-\tau_i}^{x} K\Big(x, \rho_0 y(x), \rho_1 y(x - \tau_i), \rho_2 y'(x - \tau_i)\Big) dx \right) \Bigg| \le L \Big| y^*(x) - y(x) \Big|. \end{aligned} \tag{19}$$

After subtracting the approximate solution, $y(x)$, from the exact solution, $y^*(x)$, where, $y^*_{n+1} - y_{n+1}$, $y^*_{n+2} - y_{n+2}$, $y^*_n - y_n$, ...are denoted as d_{n+1}, d_{n+2}, d_n, ...respectively. Assumed that there exists a bound, B, such as,

$$\left(1 - \frac{1}{6}hL\right)|d_{n+1}| \leq \left(1 + \frac{1}{6}hL\right)|d_n| + \frac{4}{6}hL|d_{n+\frac{1}{2}}| - \frac{1}{2880}h^5 B,$$

$$\left(1 - \frac{10}{27}hL\right)|d_{n+2}| \leq hL|d_n| + hL|d_{n+1}| + \frac{16}{27}hL|d_{n+\frac{1}{2}}| + \frac{1}{27}hL|d_{n-1}| - \frac{1}{40}h^5 B.$$

Then,

$$|d_{n+1}| \leq \frac{\left(1 + \frac{1}{6}hL\right)}{\left(1 - \frac{1}{6}hL\right)}|d_n| + \frac{\frac{4}{6}hL}{\left(1 - \frac{1}{6}hL\right)}|d_{n+\frac{1}{2}}| - \frac{1}{2880}h^5 B,$$

$$|d_{n+2}| \leq \frac{1}{\left(1 - \frac{10}{27}hL\right)}|d_n| + \frac{1hL}{\left(1 - \frac{10}{27}hL\right)}|d_{n+1}| + \frac{\frac{16}{27}hL}{\left(1 - \frac{10}{27}hL\right)}|d_{n+\frac{1}{2}}| + \frac{\frac{1}{27}hL}{\left(1 - \frac{10}{27}hL\right)}|d_{n-1}| - \frac{1}{40}h^5 B.$$

As the steplength tends to approach zero, hence,

$$y^*_{n+1} - y_{n+1} = y^*_n - y_n.$$

Based on the above interpretation, the proposed block method's convergence has been demonstrated.

4 Stability Analysis of Method

The stability properties of 1OBM4 were investigated in this study. [6] first introduced the analysis, and the test equation for the linear first-order NDVIDE with constant delay is provided by,

$$y'(x) = \xi y(x-\tau) + \nu \int_0^{x-\tau} y(u)du + \eta y'(x-\tau), \tag{20}$$

while the linear first-order NDVIDE with proportional delay is given by,

$$y'(x) = \xi y(qx) + \nu \int_0^{qx} y(u)du + \eta y'(qx). \tag{21}$$

Assume for the purpose of convenience that $x - \tau = qx = mh$, $(m \in I)$ and $y(x-\tau) = y(qx) = y_{nr}$, [21]. The corrector multistep formula for 1OBM4 is rearranged as shown below,

$$\sum_{j=0}^{2} A_j Y_{N+j} = h \sum_{j=0}^{2} B_j F_{N+j}. \tag{22}$$

A stability area is built to identify any appropriate steplength to be utilized in the mathematical calculation of the NDVIDE. By substituting the test equation into equation (22) above yields to,

$$\sum_{j=0}^{2} A_j Y_{N+j} = h \sum_{j=0}^{2} B_j \left(\xi Y_{N+j-m} + \nu \int_0^{N+j-m} Y(u)du + \eta Y'_{N+j-m}\right). \tag{23}$$

The quadrature rule adapted in obtaining the stability region is Simpson's rule where it is applied to each subinterval, with the results being summed to produce an approximation for the integral over the entire interval. This sort of approach is named as the composite Simpson's rule,

$$\int_0^x y(u)du = h\left(\frac{1}{3}Y_{N-2} + \frac{4}{3}Y_{N-1} + \frac{1}{3}Y_N\right), \tag{24}$$

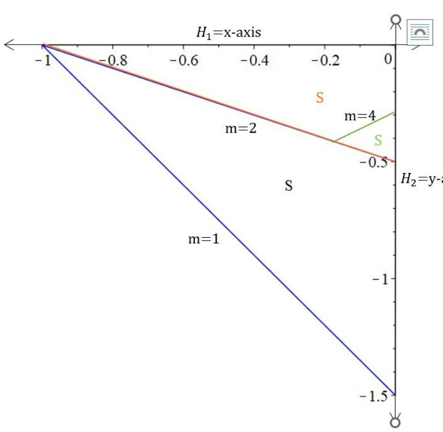
will be applied in the equation (23). After implementing the above requirements, and replacing $H_1 = \eta h$ and $H_2 = \nu h^2$, the stability polynomial obtained is as follows,

$$\begin{aligned}\pi(H_1, H_2; t) = \Bigg|&\left(A_2 - \frac{1}{3}H_2B_2\right)t^{r+2} + \left(A_1 - \frac{1}{3}H_2B_1 - \frac{4}{3}H_2B_2\right)t^{r+1} \\ &+ \left(A_0 - \frac{1}{3}H_2B_0 - \frac{4}{3}H_2B_1 - \frac{1}{3}H_2B_2\right)t^r + \left(-\frac{4}{3}H_2B_0 - \frac{1}{3}H_2B_1\right)t^{r-1} \\ &+ \left(-\frac{1}{3}H_2B_0\right)t^{r-2} + \left(-H_1B_2 - \eta B_2\right)t^2 + \left(-H_1B_1 - \eta B_1\right)t^1 \\ &+ \left(-H_1B_0 - \eta B_0\right)t^0\Bigg|,\end{aligned} \tag{25}$$

while,

$$\begin{aligned}\pi(H_1, H_2; t) = \Bigg|&\left(A_2 - \frac{1}{3}H_2B_2\right)t^{r+2} + \left(A_1 - \frac{1}{3}H_2B_1 - \frac{4}{3}H_2B_2\right)t^{r+1} \\ &+ \left(A_0 - \frac{1}{3}H_2B_0 - \frac{4}{3}H_2B_1 - \frac{1}{3}H_2B_2\right)t^r + \left(-\frac{4}{3}H_2B_0 - \frac{1}{3}H_2B_1\right)t^{r-1} \\ &+ \left(-\frac{1}{3}H_2B_0\right)t^{r-2} + \left(-H_1B_2 - \eta B_2\right)t^{3r} + \left(-H_1B_1 - \eta B_1\right)t^{2r} \\ &+ \left(-H_1B_0 - \eta B_0\right)t^0\Bigg|,\end{aligned} \tag{26}$$

for both constant and proportional delays respectively. By letting $\eta = 1$, the region of stability for the 1OBM4 is depicted in the figure (for constant delay and proportional delay) below,



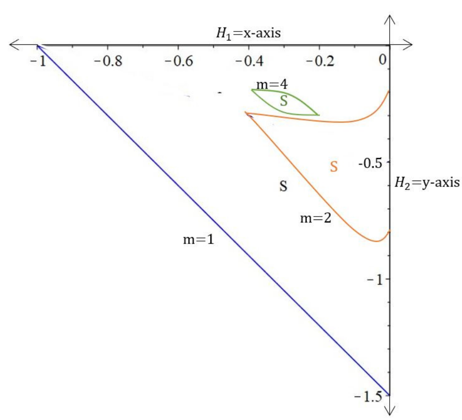


Figure 2: Areas of numerical stability for 1OBM4 with various $m = 1; 2; 4$ values. According to [11], the region diminishes as m increases. As the step size, h, decreases, the value of m increases, $(\frac{\tau}{h} = m)$, where $\tau = 1$. As m increases, the stability areas get progressively smaller. The method is proved to be stable inside of its shaded region.

where H_1 is the $x-$axis. As shown above, the first region is obtained from constant delay test equation eq.(20), while the second region is obtained from proportional delay test equation eq.(21). It is proven that the different values of m applied will affect the size of the stability region.

Definition 4.1. *If all roots of the stability polynomial satisfy $|r_s| < 1$, $s = 1, 2, \ldots, k$, the hybrid multistep approach is said to be absolutely stable and to be absolutely unstable if it is otherwise.* [18].

The stability regions are established in $(H_1 - H_2)$ plane by substituting $r = 1$, -1 and $r = \cos\theta + i\sin\theta$, $0 \geq \theta \geq 2\pi$ in the stability polynomial obtained. For $r = \cos\theta + i\sin\theta$, the real and imaginary parts are separated and solved simultaneously to obtain the points in the regions. Since the set of all roots in the stability polynomial satisfy $|r_s| < 1$, $s = 1, 2, \ldots, k$, then the stability regions obtained are absolutely stable (the numerical solution decays to zero). According to [24],

Definition 4.2. *Consider a stable method concerning the test equation and an interpolant y has a constant $m = 1$, the resulting VDIDE method is GPN-stable.*

From the above definitions, it is clear that any stability test that is concerned with the value of m, is named GPN-stable while the region satisfies $|r_s| < 1$, $s = 1, 2, \ldots, k$, for the roots of the stability polynomial is called an absolute stability region. Some of the authors that have been discussing on GPN stability region are [24, 7] and [34]. Hence, the name of stability regions obtained for 1OBM4 is GPN-stable with absolute stability regions.

5 Implementation of Method

5.1 Implementation of solving NDVIDEs for constant delay

Let's consider the constant type of NDVIDE as in equation (1). The constant type of NDVIDE has been solved by implementing 1OBM4 where two approximations including two off-point are estimated in one block using the constant step size technique. Additionally, the position of delays needs to be known first before preceding the calculation of the proposed method. Two functions of delays which are the delay terms itself, $y(x-\tau)$ and its derivatives, $y'(x-\tau)$ are taken into consideration. The delay derivative has been solved in different way by other authors such as the use of the first derivative strategy. In this research, $y'(x-\tau)$ will be evaluated by applying the backward and forward divided difference formula. The divided difference formula of order 4 is denoted as below,

$$\begin{aligned} y'_n =& \frac{-2y_n + 9y_{n-1} - 18y_{n-2} + 11y_{n-3}}{6h} (BDF4), \\ y'_n =& \frac{-11y_n + 18y_{n+1} - 9y_{n+2} + 2y_{n+3}}{6h} (FDF4). \end{aligned} \tag{27}$$

As for the value of the delay itself, τ_i, the initial function given in (1) will be applied if $x - \tau_i < a$. Otherwise, a Lagrange interpolation polynomial will be used if $x - \tau_i \geq a$. But, a constant delay case usually does not involve the evaluation of Lagrange as the delay terms will fall perfectly on the previous approximate values. The location of the delay must be discovered first in order to find both solutions of $y(x-\tau)$ and $y'(x-\tau)$. The integration part is solved by applying the composite Simpson quadrature rule. Simpson's rule is applied to each subinterval, summing the results to obtain an approximation for the integral over the entire This type of approach is called the composite Simpson's rule. Suppose that the interval $[a, b]$ is divided into n subintervals, where n is an even number. Then the composite Simpson's rule is given by,

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{3}\sum_{j=1}^{n/2}\Big[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\Big],\\ &= \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n)\right],\end{aligned} \tag{28}$$

where $x_j = a + jh$ for $j = 0, 1, \ldots, n-1, n$, with $h = \frac{b-a}{n}$ in particular, $x_0 = a$ and $x_n = b$. Before implementing 1OBM4, the initial solutions for the proposed methods need to be considered as they are both multistep block methods which do not stand on their own. Four initial solutions need to be approximated first for 1OBM4 since the predictor formulae are of order 4. A Runge-Kutta of order 4 (RK4) has been chosen to evaluate the initial solutions for the proposed method before applying 1OBM4. The formula of RK4 is shown as below,

$$\begin{aligned}y_{n+1} &= y_n + \frac{h}{6}\Bigg(k_1 + 2k_2 + 2k_3 + k_4\Bigg),\\ k_1 &= f\Bigg(x_n, y_n + \int_{n-m}^{n} K\Big(x_n, y_{n-m}, y'_{n-m}\Big)\Bigg),\\ k_2 &= f\Bigg(x_n + \frac{h}{2}, (y_n + \frac{h}{2}k_1) + \int_{n-m}^{n} K\Big(x_n + \frac{h}{2}, y_{n-m} + \frac{h}{2}k_1, y'_{n-m} + \frac{h}{2}k_1\Big)\Bigg),\\ k_3 &= f\Bigg(x_n + \frac{h}{2}, (y_n + \frac{h}{2}k_2) + \int_{n-m}^{n} K\Big(x_n + \frac{h}{2}, y_{n-m} + \frac{h}{2}k_2, y'_{n-m} + \frac{h}{2}k_2\Big)\Bigg),\\ k_4 &= f\Bigg(x_n + h, (y_n + hk_3) + \int_{n-m}^{n} K\Big(x_n + h, y_{n-m} + hk_3, y'_{n-m} + hk_3\Big)\Bigg).\end{aligned} \tag{29}$$

The numerical result will be computed by C programme with a constant step size. The algorithm for 1OBM4 is illustrated in detail on how to handle the integral part, the delay term, and its derivative. The following notations are used in the program :

a	Initial value
b	End value
h	step size
N	Number of iteration
y_0	Initial solution
$y'(x-\tau)$	Delay derivative

Algorithm of 1OBM4

Step 1 : All values given in equation, $x_0 = a, x_n = b, h, N, y_0, y'(x-\tau) \leq a$ are set.
Step 2 : The volterra integro-differential equation of neutral delay is defined:
$y'(x) = f\big(x, \rho_0 y(x)\big) + \int_{x-\tau_i}^{x} K\Big(x, \rho_0 y(x), \rho_1 y(x-\tau_i), \rho_2 y'(x-\tau_i)\Big).$
Step 3 : If $x - \tau \leq a$, then the original function given is used.
Step 4 : If $x - \tau \geq a$, then the delay terms are solved by applying any associated prior solution.
Step 5 : Backward or forward divided difference formulae are applied to find $y'(x-\tau)$.
Step 6 : Composite Simpson is applied to approximate the integral part.
Step 7 : For $n = 0, 1$,
The initial solution is computed by applying RK4 denoted in equation (29).
Step 8 : For $n = 2, 4, 6, \ldots$
Approximate NVDIDE by using the proposed method, 1OBM4.
Step 9 : Maximum and average error, total steps taken, function calls evaluatedand time consumed is calculated computationally.
Step 10 : Stop.

5.2 Implementation of solving NDVIDEs for proportional delay

Let's consider the pantograph type of NDVIDE, as mentioned in equation (2) previously. NDVIDE with pantograph equation will be solved by implementing 1OBM4 where the solutions will be evaluated by approximating two iterations in one block including the off-point. The positions of the delay terms and its derivative need to be determined whether those delays are going to be estimated using Lagrange interpolation polynomial, any additional derived method or only applying the previously estimated solutions. A constant step size technique is applied in approximating each iteration for the derived implicit hybrid method. Before implementing 1OBM4, an implicit Euler method is applied to find the solutions of the initial values. A second-order Euler method is chosen to be applied to the first few iterations since pantograph type of NDVIDE has a unique behavior. The formula of the implicit Euler is shown below,

$$
\begin{aligned}
y_{n+1}^{p} &= y_n + hf\left(x_n, y_n + \int_{n-m}^{n} K\big(x_n, y_{n-m}, y'_{n-m}\big)\right), \\
y_{n+1}^{c} &= y_n + \frac{h}{2}\left[f^{p}\left(x_n, y_n + \int_{n-m}^{n} K\big(x_n, y_{n-m}, y'_{n-m}\big)\right)\right. \\
&\qquad \left. + f\left(x_n, y_n + \int_{n-m}^{n} K\big(x_n, y_{n-m}, y'_{n-m}\big)\right)\right].
\end{aligned} \tag{30}
$$

Pantograph equation is usually estimated by Lagrange interpolation polynomial which needs a number of points in order to achieve higher order polynomial and obtain great accuracy in the approximation. The problem arises in pantograph equation as the previous points on the interval are inadequate for higher order Lagrange. The behavior of pantograph equation has made it inaccurate if being solved using any higher-order method directly. Thus, the reason explained that the first few iterations of pantograph equation need to be handled carefully using lower order method since it can cope well with lower-order Lagrange interpolation polynomials. The formula of the

Lagrange interpolation polynomial is shown as follows,

$$P(x) = L_{n,0}(x)f(x_0) + \ldots + L_{n,n}(x)f(x_n),$$
$$= \sum_{k=0}^{n} f(x_k)L_{n,k}(x),$$

where,

$$L_{n,k}(x) = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)},$$
where $k = 0, 1, \ldots, n$.

Another approach to estimating the proportional delay term is by applying an additional derived method. This is a helpful approach since the delay term will usually fall exactly on the point in the derived formula. The additional formula is given by,

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2}\left[f\left(x_n, y_n + \int_{n-m}^{n} K\Big(x_n, y_{n-m}, y'_{n-m}\Big)\right)\right].$$

Then, the proposed block method, 1OBM4 will be applied in order to continue the calculations. When solving the pantograph equation, the method uses fewer function calls and total steps than when solving the constant type. This is due to the application of an implicit Euler, which does not require many steps and functions in its calculation. As mentioned in the constant delay section, the derivative of the delay term, $y'(qx)$ is solved by implementing both backward and forward divided difference formulas in (27). The integral part will be solved by applying Composite Simpson's rule in (28). The idea of applying the composite quadrature formula is taken from [4] where the integration part is solved by using Boole's quadrature rule. The algorithms for the implicit hybrid multistep block method are constructed using C language with a constant step size. The numerical result obtained has shown the applicability and efficiency of both methods in solving proportional types of NDVIDE. The algorithm for 1OBM4 in solving the pantograph equation is shown below,

Algorithm of 1OBM4

Step 1 : All values given in equation, $x_0 = a, x_n = b, h, N, y_0, y'(qx) \leq a$ are set.
Step 2 : The volterra integro-differential equation of neutral delay is defined:
$y'(x) = f(x, \rho_0 y(x)) + \int_{qx}^{x} K\Big(x, \rho_0 y(x), \rho_1 y(qx), \rho_2 y'(qx)\Big)$.
Step 3 : If $qx \leq a$, then the original function given is used.
Step 4 : If $qx \geq a$, then the delay terms are solved by applying Lagrange interpolation polynomial, the additional method or any associated prior solution.
Step 5 : Backward or forward divided difference formulae are applied to find $y'(qx)$.
Step 6 : Composite Simpson is applied to approximate the integral part.
Step 7 : For $n = 0, 1$,
The initial solution is computed by applying implicit Euler denoted in equation (30).
Step 8 : For $n = 2, 4, 6, \ldots,$
Approximate NVDIDE by using the proposed method, 1OBM4.
Step 9 : Maximum and average error, total steps taken, function calls evaluated and time consumed is calculated computationally.
Step 10 : Stop.

Unlike constant NDVIDE, the pantograph equation needs a different implementation which has been stated in Step 4 where Lagrange interpolation polynomial or additional methods are involved. The implicit Euler of order two is chosen to be applied as the initial solution since it has the same order as Lagrange for the starting point. After completing the estimation for all initial solutions required, the proposed method 1OBM4 is then applied.

6 Numerical Results

In this segment, five NDVIDE with constant or proportional delay problems have been addressed by employing 1OBM4. Example 6.1, 6.2 and 6.3 are taken from [31], [19] and [10], respectively, known as constant NDVIDE problems. For the proportional delay, Example 6.4 is taken from [30]. The numerical results demonstrate that 1OBM4 outperforms the other methods in terms of total steps, function calls, efficiency, and accuracy. Tables 1-4 consider making use of the notations provided below,

h	:	Step size.
MTD	:	Method.
FCN	:	Total function calls.
TS	:	Total steps.
MAXE	:	Maximum error.
AVERE	:	Average error.
TIME(s)	:	Time taken in second.
1OBM4	:	Two-point implicit hybrid multistep block method with one off-point (Order 4).
2PBM4	:	Two-point multistep block method from [23] (Order 4).
ABM4	:	Adam-Bashforth-Moulton method (Order 4).
RK4	:	Runge-Kutta method (Order 4).
8e-10	:	8×10^{-10}.

Example 6.1. ***Zaidan (2012), [31]*** *(Constant delay,* $\tau = 1$*)*

$$y'(x-1) = 1 - \frac{x^3}{6} + \int_0^x (x-t)y(t)dt,$$
$$y(x) = x, \qquad x \in [-1, 0].$$

Exact solution:

$$y(x) = x, \qquad x \in [0, 1].$$

Table 1: Numerical result of 1OBM4, 2PBM4, ABM4 and RK4 for Example 6.1.

h	MTD	FCN	TS	MAXE	AVERE	TIME(s)
0.1	1OBM4	15	7	1.8035e-04	1.0947e-04	0.031
	2PBM4	16	6	5.9475e-04	2.3746e-04	0.031
	ABM4	16	10	8.8359e-04	3.5425e-04	0.036
	RK4	40	10	4.3693e-03	1.3981e-03	0.048
0.01	1OBM4	60	52	9.5671e-07	3.3148e-07	0.172
	2PBM4	106	51	5.6509e-06	1.8918e-06	0.188
	ABM4	106	100	8.4765e-06	2.8469e-06	0.219
	RK4	400	100	2.9689e-04	7.6773e-05	0.234
0.001	1OBM4	510	502	9.4218e-09	3.1255e-09	1.563
	2PBM4	1006	501	5.6484e-08	1.8695e-08	1.703
	ABM4	1006	1000	8.4727e-08	2.8052e-08	1.790
	RK4	4000	1000	2.8179e-05	7.0475e-06	1.802
0.0001	1OBM4	5010	5002	9.4147e-11	3.1134e-11	9.481
	2PBM4	10006	5001	5.6484e-10	1.8675e-10	9.720
	ABM4	10006	10000	4.2366e-10	1.4007e-10	9.858
	RK4	40000	10000	2.8027e-06	6.9846e-07	10.53

Example 6.2. ***Wen and Zhou (2017), [19]*** *(Constant delay, $\tau = 1$)*

$$\frac{d}{dx}\left[y(x) - \frac{1}{10}y(x-1)\right] = -100y(x) + \frac{y^2(x-1)}{1+y^2(x-1)} + \frac{991}{10}e^{-x} - \frac{e^{-2x+2}}{1+e^{-2x+2}} + \frac{1}{10}\int_{x-1}^{x} y(u)du,$$

$$\phi(x) = e^{-x}, \qquad x \leq 0.$$

Table 2: Numerical result of 1OBM4, 2PBM4, ABM4 and RK4 for Example 6.2.

h	MTD	FCN	TS	MAXE	AVERE	TIME(s)
0.01	1OBM4	60	52	1.5653e-03	4.4154e-05	0.172
	2PBM4	106	51	1.3946e-03	6.2132e-04	0.219
	ABM4	106	100	1.3946e-03	1.1554e-04	0.188
	RK4	400	100	1.1945e-03	8.1288e-04	0.306
0.001	1OBM4	510	502	4.2913e-05	2.0503e-07	1.531
	2PBM4	1006	501	8.3171e-05	2.9851e-05	1.625
	ABM4	1006	1000	3.4982e-05	1.0216e-05	1.688
	RK4	4000	1000	2.8275e-05	2.0546e-05	1.703
0.0001	1OBM4	5010	5002	5.7309e-06	2.1361e-09	28.30
	2PBM4	10006	5001	8.3317e-06	2.6690e-06	30.89
	ABM4	10006	10000	3.2215e-06	1.0186e-06	30.10
	RK4	40000	10000	2.2676e-06	9.8865e-07	33.22

Example 6.3. ***Horvat (1999), [10]*** *(Constant delay, $\tau = 1$, $\lambda = 3$, $\mu = 1$)*

$$y'(x) = (\lambda - \mu)e^{\tau - x} - (1 + \lambda - \mu)y(x) - \lambda \int_{x-1}^{x} y(t)dt - \mu \int_{x-1}^{x} y'(t)dt,$$
$$\phi(x) = e^{-x}, \qquad x \leq 0.$$

Exact solution:

$$y(x) = e^{-x}, \qquad x \in [0, 1].$$

Table 3: Numerical result of 1OBM4, 2PBM4, ABM4 and RK4 for Example 6.3.

h	MTD	FCN	TS	MAXE	AVERE	TIME(s)
0.1	1OBM4	15	7	1.5138e-02	3.1323e-03	0.047
	2PBM4	16	6	2.5920e-02	3.5285e-03	0.068
	ABM4	16	10	2.3865e-02	1.3759e-02	0.078
	RK4	40	10	2.5920e-02	1.2274e-02	0.091
0.01	1OBM4	60	52	1.9373e-04	7.2909e-05	0.172
	2PBM4	106	51	2.7670e-04	8.6207e-05	0.266
	ABM4	106	100	1.3954e-03	7.6639e-04	0.252
	RK4	400	100	2.7670e-04	1.0316e-04	0.318
0.001	1OBM4	510	502	1.5008e-05	7.2442e-07	2.063
	2PBM4	1006	501	2.4008e-05	7.6263e-06	3.203
	ABM4	1006	1000	1.3090e-04	7.3490e-05	3.329
	RK4	4000	1000	5.7687e-06	2.8982e-06	3.540
0.0001	1OBM4	5010	5002	1.6132e-06	7.3182e-09	95.87
	2PBM4	10006	5001	2.3879e-06	7.5515e-07	99.86
	ABM4	10006	10000	1.3006e-05	7.3224e-06	106.9
	RK4	40000	10000	6.8277e-07	4.2410e-07	119.3

Example 6.4. $\left(\textit{Proportional delay, } \tau = \dfrac{x}{2}\right)$

$$y'(x) = \cos\left(\frac{x}{2}\right)y\left(\frac{x}{2}\right) + \int_{0}^{\frac{x}{2}} \Big[\sin(t)y(t) - \cos(t)y'(t)\Big]dt + \cos(x),$$
$$y(0) = 0,$$

Exact solution:

$$y(x) = \sin(x), \qquad x \in [0, 1].$$

Table 4: Numerical result of 1OBM4, 2PBM4, ABM4 and RK4 for Example 6.4.

h	MTD	FCN	TS	MAXE	AVERE	TIME(s)
0.01	1OPBM4	51	52	1.8956e-02	7.8089e-03	0.818
	2PBM4	100	51	1.9341e-02	7.8738e-03	0.869
	ABM4	100	100	9.6930e-03	5.2706e-03	0.875
	RK4	400	100	4.7846e-01	7.6494e-03	0.963
0.001	1OPBM4	501	502	1.9335e-03	7.9580e-04	3.726
	2PBM4	1000	501	1.9605e-03	8.0275e-04	4.813
	ABM4	1000	1000	9.8049e-04	5.3539e-04	5.195
	RK4	4000	1000	4.7932e-01	7.6658e-04	5.405
0.0001	1OPBM4	5001	5002	1.9373e-04	7.9730e-05	56.170
	2PBM4	10000	5001	1.9631e-04	8.0430e-05	63.120
	ABM4	10000	10000	9.8159e-05	5.3622e-05	69.364
	RK4	40000	10000	4.7942e-01	7.6675e-05	104.100

6.1 Order of convergence for 1OBM4

Other than the approximate solution in equation (15), the exact solution is also taken into consideration,

$$\begin{aligned} y^*_{n+1} &= y_n + \frac{h}{6}\Big(f_{n+1} + 4f_{n+\frac{1}{2}} + f_n\Big) - \frac{1}{2880}h^5 y^{*(5)}(\xi_n), \\ y^*_{n+2} &= y_n + \frac{h}{27}\Big(10f_{n+2} + 27f_{n+1} + 16f_{n+\frac{1}{2}} + f_{n-1}\Big) - \frac{1}{40}h^5 y^{*(5)}(\xi_n). \end{aligned} \tag{31}$$

to estimate the order of convergence for 1OBM4. The formula to estimate the order of convergence is given by,

$$q = \frac{\log\left(\frac{e_{\text{new}}}{e_{\text{old}}}\right)}{\log\left(\frac{h_{\text{new}}}{h_{\text{old}}}\right)}, \tag{32}$$

where,

$$\begin{aligned} e_{\text{new}} &= \Big|\text{exact value} - \text{approximate value with } h_{\text{new}} \text{ steplength}\Big|, \\ e_{\text{old}} &= \Big|\text{exact value} - \text{approximate value with } h_{\text{old}} \text{ steplength}\Big|, \\ h_{\text{new}} &= \text{steplength at } (i+1)^{th} \text{ stage}, \\ h_{\text{old}} &= \text{steplength at } (i)^{th} \text{ stage}. \end{aligned}$$

Therefore, the order of convergent is obtained as follows,

$$q = \frac{\log\left(\frac{1.8035e-04}{9.5671e-07}\right)}{\log\left(\frac{0.1}{0.01}\right)} = 2.2753.$$

The $e_{\text{new}} = 1.8035e-04$ and $e_{\text{old}} = 9.5671e-07$, are taken from the maximum error in Table 1 at $h = 0.1$ and $h = 0.01$ from 1OBM4 respectively. The value for the order of convergence should approach 4 since the order of the proposed method is four. Some constraints have affected the value obtained since the solutions for NDVIDE involve the delay problem. The order of the method at the beginning of the interval should be low to solve the delay since the points calculated are not enough. Thus, any higher-order method applied will cause some restrictions.

7 Discussion

In this contribution, the given examples have indicated and illustrated some roles that may be played by NDVIDE in modeling certain cell development phenomena that exhibit a time lag in responding to events. In Examples 6.1 and 6.3, it is obvious that $\tau = 1$ is a constant delay, and based on [5], any $\tau \geq 0$ defined an average time for the cell division. The proposed method has performed very well in solving NDVIDE in Examples 6.1 and 6.3 where the accuracy in maximum and average error is getting better. Hence, the method is proven to reduce any causing-delay element in cell development. As in Example 6.2, the NDVIDE denotes the cell-death rate since $-\rho_0 = -100 \leq 0$. Cell death is defined as the failure of a biological cell to perform its functions. This could be due to a natural process in which old cells die and are replaced by new ones, as in programmed cell death, or it could be due to events such as diseases, localized injury, or the death of the organism of which the cells are apart. The function reduction from 1OBM4 has increased its advantage in reducing the cell-death rate compared to other methods. Other than that, the proposed method also managed to produce accurate results (able to produce healthy cells) even in a longer time (bigger N).

Finally, from the numerical results obtained in Example 6.4, the proposed method has produced comparable results to the other two methods (2PBM4 and ABM4). Nonetheless, the time consumed in seconds (s) for the produced techniques is lesser than that of the comparison methods since the points acquired in developing the methods do not require recalculation after iterations as 2PBM4 does. The hybrid approaches have also reduced the number of steps required when compared to ABM4 and RK4 (non-block methods). In terms of function calls and time spent, the proposed methods outperformed all comparison methods.

8 Conclusion

Compared to the previous methods, the proposed method has shown to be applicable and efficient in modeling NDVIDE problems with constant or proportional delays by predicting a few points at a time and incorporating the off-point in the predictor. Since the main goal of the discussion is to demonstrate the benefits of an implicit hybrid multistep block method, all parameters in comparing block and non-block methods must be addressed.

Acknowledgement All authors gratefully acknowledge the financial support of the Fundamental Research Grant Scheme (project code: FRGS/1/2020/STG06/UPM/01/1) from the Ministry of Higher Education Malaysia. The authors are also thankful to the referees for their useful comments and suggestions.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] A. Ackleh, E. Allen, R. B. Kearfott & P. Seshaiyer (2009). *Classical and modern numerical analysis: Theory, methods and practice*. CRC Press, United Kingdom.

[2] B. Alfredo & M. Zennaro (2013). *Numerical methods for delay differential equations*. Oxford University Press, New York.

[3] Y. Altun (2021). Asymptotic behaviors of the solutions of neutral type Volterra integro-differential equations and some numerical solutions via differential transform method. *MANAS Journal of Engineering*, *9*(S), 49–57. https://doi.org/10.51354/mjen.878066.

[4] N. A. Baharum, Z. A. Majid & N. Senu (2022). Boole's strategy in multistep block method for Volterra integro-differential equation. *Malaysian Journal of Mathematical Sciences*, *16*(2), 237–256. https://doi.org/10.47836/mjms.16.2.05.

[5] C. T. H. Baker, G. Bocharov & F. A. Rihan (2008). *Neutral delay differential equations in the modeling of cell growth* (*Applied Mathematics Group Research Report, 2008: 1*). University of Chester, Chester, United Kingdom.

[6] A. Bellen, Z. Jackiewicz & M. Zennaro (1988). Stability analysis of one-step methods for neutral delay-differential equations. *Numerische Mathematik*, *52*(6), 605–619. https://doi.org/10.1007/BF01395814.

[7] H. Brunner (1996). Open problems in the discretization of Volterra integral equations: The discretization of Volterra integral equations. *Numerical Functional Analysis and Optimization*, *17*(7–8), 717–736. https://doi.org/10.1080/01630569608816721.

[8] H. Brunner (2007). High-order collocation methods for singular Volterra functional equations of neutral type. *Applied Numerical Mathematics*, *57*(5–7), 533–548. https://doi.org/10.1016/j.apnum.2006.07.006.

[9] W. Enright & M. Hu (1997). Continuous Runge-Kutta methods for neutral Volterra integro-differential equations with delay. *Applied Numerical Mathematics*, *24*(2-3), 175–190. https://doi.org/10.1016/S0168-9274(97)00019-6.

[10] V. Horvat (1999). On polynomial spline collocation methods for neutral Volterra integro-differential equations with delay arguments. In *Proceedings of the 1. Conference on Applied Mathematics and Computation*, pp. 113–128. Dubrovnik, Croatia. https://api.semanticscholar.org/CorpusID:221785689.

[11] F. Ishak, M. Suleiman & Z. A. Majid (2014). Numerical solution and stability of block method for solving functional differential equations. In *Transactions on engineering technologies: Special volume of the world congress on engineering*, pp. 597–609. Springer, Netherlands.

[12] N. I. N. Ismail, Z. A. Majid & N. Senu (2020). Solving neutral delay differential equation of pantograph type. *Malaysian Journal of Mathematical Sciences*, *14*(S), 107–121.

[13] N. I. N. Ismail, Z. A. Majid & N. Senu (2022). Numerical solution for neutral delay differential equation of constant or proportional type using hybrid block method. *Advances in Applied Mathematics and Mechanics*, *14*(5), 1138–1160. https://doi.org/10.4208/aamm.OA-2020-0054.

[14] N. I. N. Ismail, Z. A. Majid, N. Senu & N. Wahi (2021). Numerical method in solving neutral and retarded Volterra delay integro-differential equations. In *Journal of Physics: Conference Series*, pp. 012033. IOP Science,.

[15] Z. Jackiewicz (1981). The numerical solution of Volterra functional differential equations of neutral type. *SIAM Journal on Numerical Analysis*, *18*(4), 615–626. https://doi.org/10.1137/0718040.

[16] S. Jator (2010). On the hybrid method with three off-step points for initial value problems. *International Journal of Applied and Computational Mathematics*, *41*(1), 110–118. https://doi.org/10.1080/00207390903189203.

[17] H. Laib, A. Bellour & A. Boulmerka (2022). Taylor collocation method for high-order neutral delay Volterra integro-differential equations. *Journal of Innovative Applied Mathematics and Computational Sciences*, *2*(1), 53–77. https://doi.org/10.58205/jiamcs.v2i1.10.

[18] J. D. Lambert (1973). *Computational methods in ordinary differential equations*. John Wiley & Sons, London.

[19] W. Liping & Z. Yan (2017). Convergence of one-leg methods for neutral delay integro-differential equations. *Journal of Computational and Applied Mathematics*, *317*, 432–446. https://doi.org/10.1016/j.cam.2016.12.011.

[20] Z. A. Majid & M. Suleiman (2011). Predictor-corrector block iteration method for solving ordinary differential equations. *Sains Malaysiana*, *40*(6), 659–664.

[21] F. A. Rihan, E. Doha, M. I. Hassan & N. Kamel (2009). Numerical treatments for Volterra delay integro-differential equations. *Computational Methods in Applied Mathematics*, *9*(3), 292–318. https://doi.org/10.2478/cmam-2009-0018.

[22] S. Sedaghat, Y. Ordokhani & M. Dehghan (2014). On spectral method for Volterra functional integro-differential equations of neutral type. *Numerical Functional Analysis and Optimization*, *35*(2), 223–239. https://doi.org/10.1080/01630563.2013.867189.

[23] H. Y. Seong (2016). *Multistep blocks methods for solving higher order delay differential equation*. PhD thesis, Universiti Putra Malaysia, Malaysia.

[24] L. Torelli (1991). A sufficient condition for GPN-stability for delay differential equations. *Numerische Mathematik*, *59*(1), 311–320. https://doi.org/10.1007/BF01385783.

[25] C. Tunç (2018). Asymptotic stability and boundedness criteria for nonlinear retarded Volterra integro-differential equations. *Journal of King Saud University-Science*, *30*(4), 531–536. https://doi.org/10.1016/j.jksus.2017.05.003.

[26] C. Tunç (2021). On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation. *Revista de la Real Academia de Ciencias Exactas, FÃsicas y Naturales. Serie A. MatemÃ¡ticas*, *115*(3), 115. https://doi.org/10.1007/s13398-021-01058-8.

[27] V. Vijayakumar (2018). Approximate controllability results for abstract neutral integro-differential inclusions with infinite delay in Hilbert spaces. *IMA Journal of Mathematical Control and Information*, *35*(1), 297–314. https://doi.org/10.1093/imamci/dnw049.

[28] A. M. Wazwaz (2011). *Linear and nonlinear integral equations*. Springer, Netherlands.

[29] S. Wu & S. Gan (2008). Analytical and numerical stability of neutral delay integro-differential equations and neutral delay partial differential. *Computers & Mathematics with Applications*, *55*(11), 2426–2443. https://doi.org/10.1016/j.camwa.2007.08.045.

[30] Ş. Yüzbaşı (2014). Laguerre approach for solving pantograph-type Volterra integro-differential equations. *Applied Mathematics and Computation*, *232*, 1183–1199. https://doi.org/10.1016/j.amc.2014.01.075.

[31] L. I. Zaidan (2012). Solving linear delay Volterra integro-differential equations by using Galerkin's method with Bernstein polynomial. *Journal of University of Babylon*, *20*(5), 1405–1413.

[32] C. Zhang, F. Deng, H. Mo & H. Ren (2019). Dissipativity and stability of a class of nonlinear multiple time delay integro-differential equations. *Journal of Nonlinear Sciences & Applications*, *12*(6), 363–375. https://dx.doi.org/10.22436/jnsa.012.06.03.

[33] C. Zhang & V. Stefan (2008). Stability criteria for exact and discrete solutions of neutral multidelay-integro-differential equations. *Advances in Computational Mathematics*, *28*(4), 383–399. https://doi.org/10.1007/s10444-007-9037-4.

[34] G. Zhang & A. Xiao (2016). Exact and numerical stability analysis of reaction-diffusion equations with distributed delays. *Frontiers of Mathematics in China*, *11*(1), 189–205. https://doi.org/10.1007/s11464-015-0506-7.

[35] J. Zhao, Y. Xu & M. Liu (2005). Stability analysis of numerical methods for linear neutral Volterra delay integro-differential system. *Applied Mathematics and Computation*, *167*(2), 1062–1079. https://doi.org/10.1016/j.amc.2004.08.003.